

# Example Random Processes

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Based on  
Probability, Random Variables and Random Signal Principles,  
P.Z. Peebles,Jr. and B. Shi

# Outline

1 Gaussian Random Processes

2 Poisson Random Process

# Gaussian Random Process

$N$  Gaussian random variables

## Definition

$$f_X(x_1, \dots, x_N; t_1, \dots, t_N) = \frac{\exp \left\{ -(1/2) [x - \bar{X}]^t [C_X]^{-1} [x - \bar{X}] \right\}}{\sqrt{(2\pi)^N |[C_X]|}}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} \quad \bar{X} = \begin{bmatrix} \bar{X}_1 \\ \bar{X}_2 \\ \vdots \\ \bar{X}_N \end{bmatrix} \quad [x - \bar{X}] = \begin{bmatrix} x_1 - \bar{X}_1 \\ x_2 - \bar{X}_2 \\ \vdots \\ x_N - \bar{X}_N \end{bmatrix}$$

# The Covariance Matrix (1)

$N$  Gaussian random variables

## Definition

$$\bar{X}_i = E[X_i] = E[X(t_i)]$$

$$\bar{X} = \begin{bmatrix} \bar{X}_1 \\ \bar{X}_2 \\ \vdots \\ \bar{X}_N \end{bmatrix} = \begin{bmatrix} E[X_1] \\ E[X_2] \\ \vdots \\ E[X_N] \end{bmatrix} = \begin{bmatrix} E[X(t_1)] \\ E[X(t_2)] \\ \vdots \\ E[X(t_N)] \end{bmatrix}$$

# The Covariance Matrix (2)

$N$  Gaussian random variables

## Definition

$$\begin{aligned}C_{ik} &= C_{x_i x_k} = E[(x_i - \bar{x}_i)(x_k - \bar{x}_k)] \\&= E[(x(t_i) - E[x(t_i)])(x(t_k) - E[x(t_k)])]\end{aligned}$$

$$\begin{aligned}C_{ik} &= C_{x_i x_k} = C_{xx}(t_i, t_k) \\&= R_{xx}(t_i, t_k) - E[x(t_i)]E[x(t_k)]\end{aligned}$$

# Stationary Gaussian Process

$N$  Gaussian random variables

## Definition

$$\bar{X}_i = E[X_i] = E[X(t_i)] = \bar{X} = \text{const}$$

$$C_{XX}(t_i, t_k) = C_{XX}(t_k - t_i)$$

$$R_{XX}(t_i, t_k) = R_{XX}(t_k - t_i)$$

# Jointly Gaussian Process

$N$  Gaussian random variables

## Definition

the two random processes  $X(t)$  and  $Y(t)$   
are **jointly Gaussian** if the random variables

$X(t_1), \dots, X(t_N)$  at times  $t_1, \dots, t_N$  for  $X(t)$  and

$Y(t'_1), \dots, Y(t'_M)$  at times  $t'_1, \dots, t'_M$  for  $Y(t)$

are **jointly gaussian** for any  $N$ ,  $t_1, \dots, t_N$ , and  $M$ ,  $t'_1, \dots, t'_M$



# Stationary Gaussian Markov Process

$N$  Gaussian random variables

## Definition

$$C_{xx}(\tau) = \sigma^2 e^{-\beta|\tau|}$$

$$C_{xx}[k] = \sigma^2 a^{-|k|}$$

$$a = e^{\beta T_s}$$

# Poisson Random Process

$N$  Gaussian random variables

## Definition

$$p[X(t) = k] = \frac{(\lambda t)^k e^{-\lambda t}}{k!}, \quad k = 0, 1, 2, \dots$$

$$f_X(x) = \sum_{k=0}^{\infty} \frac{(\lambda t)^k e^{-\lambda t}}{k!} \delta(x - k)$$

# Poisson Random Process - mean and 2nd moment

$N$  Gaussian random variables

## Definition

$$\begin{aligned} E[X(t)] &= \int_{-\infty}^{\infty} x f_X(x) dx = \int_{-\infty}^{\infty} x \sum_{k=0}^{\infty} \frac{(\lambda t)^k e^{-\lambda t}}{k!} \delta(x - k) dx \\ &= \sum_{k=0}^{\infty} \frac{k(\lambda t)^k e^{-\lambda t}}{k!} = \lambda t \end{aligned}$$

$$\begin{aligned} E[X^2(t)] &= \int_{-\infty}^{\infty} x^2 f_X(x) dx = \int_{-\infty}^{\infty} x^2 \sum_{k=0}^{\infty} \frac{(\lambda t)^k e^{-\lambda t}}{k!} \delta(x - k) dx \\ &= \sum_{k=0}^{\infty} \frac{k^2 (\lambda t)^k e^{-\lambda t}}{k!} = \lambda t (1 + \lambda t) \end{aligned}$$

# Poisson Random Process - joint probability density (1)

$N$  Gaussian random variables

## Definition

$$P[X(t_1) = k_1] = \frac{(\lambda t_1)^{k_1} e^{-\lambda t_1}}{k_1!} \quad k_1 = 0, 1, 2, \dots$$

$$P[X(t_2) = k_2 | X(t_1) = k_1] = \frac{[\lambda(t_2 - t_1)]^{k_2 - k_1} e^{-\lambda(t_2 - t_1)}}{(k_2 - k_1)!}$$

$$P(k_1, k_2) = P[X(t_2) = k_2 | X(t_1) = k_1] \cdot P[X(t_1) = k_1]$$

## Poisson Random Process - joint probability density (2)

 $N$  Gaussian random variables

## Definition

$$P(k_1, k_2) = P[X(t_2) = k_2 | X(t_1) = k_1] \cdot P[X(t_1) = k_1]$$

$$= \frac{(\lambda t_1)^{k_1} [\lambda(t_2 - t_1)]^{k_2 - k_1} e^{-\lambda t_2}}{k_1! (k_2 - k_1)!}$$

$$f_X(x_1, x_2) = \sum_{k_1=0}^{\infty} \sum_{k_2=k_1}^{\infty} P(k_1, k_2) \delta(x_1 - k_1) \delta(x_2 - k_2)$$

# Counting Process (1)

$N$  Gaussian random variables

## Definition

A **random process** is called a **counting process** if

- 1 the possible states are the non-negative integers.
- 2 for each **state**  $i$  the only possible transitions are  $i \rightarrow i, i \rightarrow i+1, i \rightarrow i+2, \dots$

<http://individual.utoronto.ca/ranodya/7P1.html>

# Counting Process (2)

$N$  Gaussian random variables

## Definition

A **counting process** is said to be a **Bernoulli counting process** if

- 1 the **number of successes** that can occur in each frame is either 0 or 1
- 2 the **probability**  $p$  that a success occurs during any frame is the same for all frames
- 3 **successes** in non-overlapping frames are independent of one another

<http://individual.utoronto.ca/ranodya/7P1.html>

# Counting Process (3)

$N$  Gaussian random variables

## Definition

Let  $X[n]$  denote the total number of successes  
in a **Bernoulli counting process**

at the end of the  $n$ -th frame,  $n = 1, 2, 3, \dots$

Let the initial state be  $X[0] = 0$ .

The **probability distribution** of  $X[n]$  is the **binomial**

$$P(X[n] = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

<http://individual.utoronto.ca/ranodya/7P1.html>



# Counting Process (4)

$N$  Gaussian random variables

## Definition

If  $I$  denotes the expected number of successes in a unit of time, and  $n$  is the number of frames in this unit of time, then

$$p = I/n$$

The constant  $I$  is called the rate of success, and is estimated by

$$\hat{I} = \text{Number of successes in } t \text{ units of time} / t$$

<http://individual.utoronto.ca/ranodya/7P1.html>

# Counting Process (5)

$N$  Gaussian random variables

## Definition

If there are  $n$  frames in the unit of time and we let the **frame length**, denoted by  $D$ , be expressed in the same unit as  $I$  (seconds, minutes, hours, etc.), then

$$D = 1/n$$

Moreover,

$$p = ID$$

<http://individual.utoronto.ca/ranodya/7P1.html>

# Bernoulli Random Process (1)

$N$  Gaussian random variables

## Definition

A **Bernoulli process** is a finite or infinite sequence of independent random variables  $I[1], I[2], I[3], \dots$ , such that for each  $n$ , the value of  $I[n]$  is either 0 or 1;

for all values of  $n$ , the probability  $p$  that  $I[n] = 1$  is the same.

In other words, a **Bernoulli process** is a sequence of independent identically distributed **Bernoulli trials**.

Independence of the trials implies that the process is memoryless.

# Bernoulli Random Process (2)

$N$  Gaussian random variables

## Definition

the **Bernoulli random process** at sample index  $n$  is  $I[n]$   
the number of events that have occurred  
after sample index 0 and up to  $n$

$$X[n] = \sum_{m=1}^n I[m]$$

the **binomial counting process** is an example of what is called a **sum process**, since it can be obtained by summing the values of another random process

## Bernoulli Random Process (3)

 $N$  Gaussian random variables

## Definition

the density function for  $X[n]$  is represented by a binomial density function

$$f_X(x) = \sum_{k=0}^n P(k) \delta(x - k)$$

$$P(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

the mean and the variance of the binomial counting process

$$E[X[n]] = np$$

$$\text{Var}[X[n]] = np(1-p)$$

# Binomial Counting Process

$N$  Gaussian random variables

## Definition

$$f_X(x_1, x_2) = \sum_{k_1=0}^{n_1} \sum_{k_2=k_1}^{n_2} P(k_1, k_2) \delta(x_1 - k_1) \delta(x_2 - k_2)$$

$$P(k_1, k_2) = P[X[n_1] = k_1, X[n_2] = k_2]$$

$$= \binom{n_2 - n_1}{k_2 - k_1} \binom{n_1}{k_1} p^{k_2} (1-p)^{n_2 - k_2}$$

$$P(k) = \frac{(np)^k e^{-np}}{k!} = \frac{(\lambda t)^k e^{-\lambda t}}{k!}$$



